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Leptonic unitarity triangles in matter

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Abstract. We present a geometric description of lepton flavor mixing and CP violation in matter by using the language of leptonic unitarity triangles. The exact analytical relations for both sides and inner angles are established between every unitarity triangle in vacuum and its effective counterpart in matter. The typical shape evolution of six triangles with the terrestrial matter density is illustrated for a realistic long-baseline neutrino oscillation experiment.

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1 Introduction

Recent solar [1], atmospheric [2], reactor (KamLAND [3] and CHOOZ [4]) and accelerator (K2K [5]) neutrino oscillation experiments have provided us with very convincing evidence that neutrinos are massive and lepton flavors are mixed. In the framework of three lepton families, the phenomena of flavor mixing and CP violation are described by a 3×3 unitary matrix V, which relates the mass eigenstates of three neutrinos (ν_1, ν_2, ν_3) to their flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{1}$$

No matter whether neutrinos are Dirac or Majorana particles, leptonic CP and T violation in normal neutrino–neutrino or antineutrino–antineutrino oscillations depends only upon a single rephasing-invariant parameter $\mathcal{J}$ [6], defined through

$$\mathrm{Im}(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}) = \mathcal{J} \sum_{\gamma,k} (\epsilon_{\alpha\beta\gamma} \epsilon_{ijk}) , \tag{2}$$

where Greek and Latin subscripts run respectively over (e, μ, τ) and $(1, 2, 3)$. A challenging task of the future long-baseline neutrino oscillation experiments is to measure $\mathcal{J}$, so as to establish the existence of CP violation in the lepton sector. In principle, a determination of $\mathcal{J}$ is also possible from the four independent moduli $|V_{\alpha i}|$ [7] whose magnitudes can be measured in some CP conserving processes.

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The unitarity of V implies that its nine matrix elements are constrained by two sets of normalization conditions and two sets of orthogonality relations:

$$\sum_i (V^*_{\alpha i} V_{\beta i}) = \delta_{\alpha\beta} , \qquad \sum_\alpha (V^*_{\alpha i} V_{\alpha j}) = \delta_{ij} . \tag{3}$$

The six orthogonality relations define six triangles in the complex plane, as first discussed in [8]. Six unitarity triangles have 18 different sides and nine different inner angles, but their areas are all identical to $\mathcal{J}/2$. So far, some particular attention has been paid to triangles $\triangle_3$ [8] and $\triangle_\tau$ [9] shown in Fig. 1. Because current experimental data indicate that V has a nearly bi-maximal mixing pattern with $|V_{e3}| \ll 1$, one can easily observe that three sides of $\triangle_3$ are comparable in magnitude; i.e., $|V_{e1}V^*_{e2}| \sim |V_{\mu 1}V^*_{\mu 2}| \sim |V_{\tau 1}V^*_{\tau 2}| \sim 0.5$. It is therefore possible to establish $\triangle_3$ and determine its three angles [7], once its three sides are measured to an acceptable degree of accuracy. In contrast, one side of $\triangle_\tau$ is much shorter than its other two sides; i.e., $|V_{e1}V^*_{\mu 1}| \sim |V_{e2}V^*_{\mu 2}| \gg |V_{e3}V^*_{\mu 3}|$. To establish $\triangle_\tau$ needs much more precise data on its three sides, which must be able to show $|V_{e1}V^*_{\mu 1}| + |V_{e3}V^*_{\mu 3}| > |V_{e2}V^*_{\mu 2}|$ or $|V_{e2}V^*_{\mu 2}| + |V_{e3}V^*_{\mu 3}| > |V_{e1}V^*_{\mu 1}|$ [9]. Such an accuracy requirement is practically impossible to be satisfied in the near future[1].

[1] To see this point more clearly, one may recall the Cabibbo–Kobayashi–Maskawa (CKM) unitarity triangle of quark flavor mixing defined by $V_{ud}V^*_{cd} + V_{us}V^*_{cs} + V_{ub}V^*_{cb} = 0$, whose two sides are much longer than the other one: $|V_{ud}V^*_{cd}| \sim |V_{us}V^*_{cs}| \gg |V_{ub}V^*_{cb}|$. Although the six CKM matrix elements associated with this triangle have all been measured to a relatively good degree of accuracy [24], it remains unable to show $|V_{ud}V^*_{cd}| +$

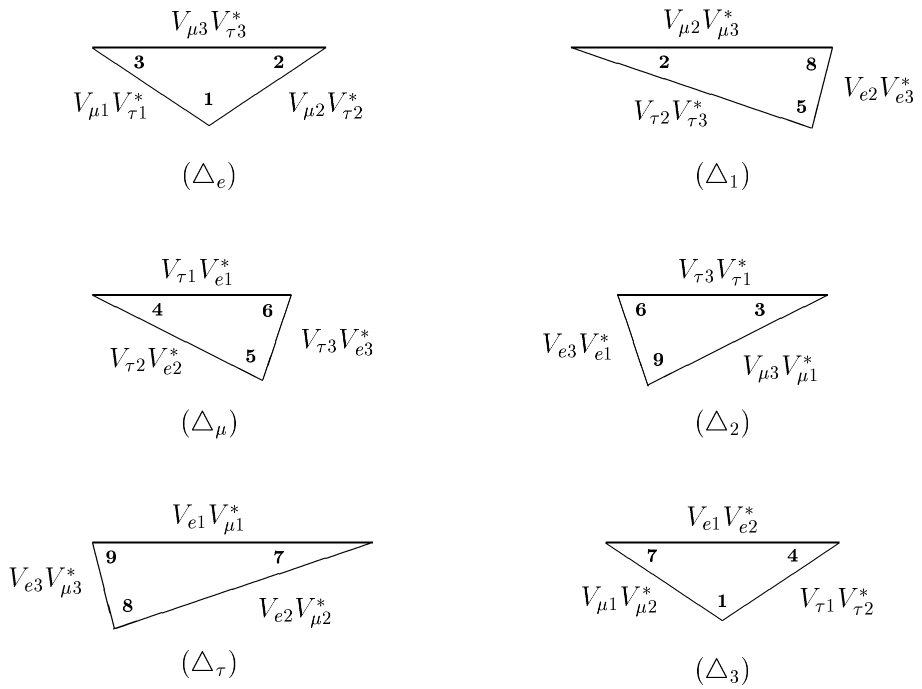

Fig. 1. Leptonic unitarity triangles in the complex plane. Each triangle is named by the index that is not manifest in its three sides

To measure $\mathcal{J}$ and $|V_{\alpha i}|$ in realistic long-baseline experiments of neutrino oscillations, the terrestrial matter effects must be taken into account [10]. The probabilities of neutrino oscillations in matter can be expressed in the same form as those in vacuum, however, if we define the *effective* neutrino masses $\tilde{m}_i$ and the *effective* lepton flavor mixing matrix $\tilde{V}$ in which the terrestrial matter effects are already included [11]. In analogy to the definition of $\mathcal{J}$ in (2), the *effective* CP violating parameter $\tilde{\mathcal{J}}$ in matter can be defined as

$$\mathrm{Im}(\tilde{V}_{\alpha i}\tilde{V}_{\beta j}\tilde{V}^*_{\alpha j}\tilde{V}^*_{\beta i}) = \tilde{\mathcal{J}}\sum_{\gamma,k}(\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}) \ , \quad (4)$$

where (α,β,γ) and (i,j,k) run respectively over (e,μ,τ) and $(1,2,3)$. One may similarly define the unitarity triangles in matter with the help of the unitarity conditions

$$\sum_i \left(\tilde{V}^*_{\alpha i}\tilde{V}_{\beta i}\right) = \delta_{\alpha\beta} \ ,$$
$$\sum_\alpha \left(\tilde{V}^*_{\alpha i}\tilde{V}_{\alpha j}\right) = \delta_{ij} \ . \quad (5)$$

For example, the *effective* counterparts of the triangles $\triangle_3$ and $\triangle_\tau$ are denoted respectively by $\tilde{\triangle}_3$ and $\tilde{\triangle}_\tau$. Because of the terrestrial matter effects, the shapes of $\tilde{\Delta}_3$ and $\tilde{\Delta}_\tau$ are likely to be dramatically different from those of Δ_3 and Δ_τ. It is then possible to have $|\tilde{V}_{e1}\tilde{V}^*_{\mu 1}| \sim |\tilde{V}_{e2}\tilde{V}^*_{\mu 2}| \sim |\tilde{V}_{e3}\tilde{V}^*_{\mu 3}|$ for some proper values of the neutrino beam energy. If this speculation is right, one will be able to calculate $\tilde{\mathcal{J}}$ by using three sides of $\tilde{\triangle}_\tau$.

$|V_{ub}V^*_{cb}| > |V_{us}V^*_{cs}|$ or $|V_{us}V^*_{cs}| + |V_{ub}V^*_{cb}| > |V_{ud}V^*_{cd}|$ – namely, the area of this triangle (or CP violation) cannot be established from its three sides at present

The purpose of this article is to carry out a systematic analysis of leptonic unitarity triangles in matter. We shall derive the exact analytical relations between $|V_{\alpha i}|^2$ and $|\tilde{V}_{\alpha i}|^2$ for a constant matter density profile. The sides of $\tilde{\triangle}_i$ (for $i=1,2,3$) and $\tilde{\triangle}_\alpha$ (for $\alpha = e,\mu,\tau$) can then be linked to those of $\triangle_i$ and $\triangle_\alpha$. The inner angles of $\tilde{\triangle}_i$ and $\tilde{\triangle}_\alpha$ will also be calculated in terms of the inner angles of $\triangle_i$ and $\triangle_\alpha$. In addition, we shall discuss the matter-modified rephasing invariants of V (such as $\tilde{\mathcal{J}}$ and the off-diagonal asymmetries of $\tilde{V}$) and illustrate the typical shape changes of $\tilde{\triangle}_i$ and $\tilde{\triangle}_\alpha$ with the neutrino beam energy in a long-baseline experiment. Our results are expected to be very useful for a complete study of lepton flavor mixing and CP violation in the era of precision measurements.

The remaining parts of this article are organized as follows. In Sect. 2, we outline the master formulas to derive $\tilde{m}_i$ and $\tilde{V}$ in terms of m_i and V. Section 3 is devoted to the calculation of $|\tilde{V}_{\alpha i}|^2$. The analytical relations between the inner angles of $\tilde{\triangle}_i$ (or $\tilde{\triangle}_\alpha$) and $\triangle_i$ (or $\triangle_\alpha$) are presented in Sect. 4. Section 5 is devoted to illustrating the shape evolution of $\tilde{\triangle}_i$ and $\tilde{\triangle}_\alpha$ with the matter density. Finally, a brief summary is given in Sect. 6.

2 Framework

In the flavor basis chosen in (1), the effective Hamiltonians responsible for the propagation of neutrinos in vacuum and in matter can respectively be written as

$$\mathcal{H} = \frac{1}{2E}\left[V\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}V^\dagger\right] \ ,$$

$$\tilde{\mathcal{H}} = \frac{1}{2E}\left[\tilde{V}\begin{pmatrix}\tilde{m}_1^2 & 0 & 0\\ 0 & \tilde{m}_2^2 & 0\\ 0 & 0 & \tilde{m}_3^2\end{pmatrix}\tilde{V}^\dagger\right] \; ; \tag{6}$$

and their difference

$$\tilde{\mathcal{H}} - \mathcal{H} = \begin{pmatrix} a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{7}$$

signifies the matter effect [12], where E denotes the neutrino beam energy, $a = \sqrt{2}G_{\rm F}N_e$ measures the charged-current contribution to the $\nu_e e^-$ forward scattering, and N_e is the background density of electrons. In writing out the expression of $\tilde{\mathcal{H}}$, we have assumed that the matter density profile is constant (namely, N_e = constant). Such an assumption is actually close to reality for most of the proposed terrestrial long-baseline neutrino oscillation experiments [13]. In order to establish the analytical relationship between $\tilde{V}_{\alpha i}$ and $V_{\alpha i}$, we define the two quantities

$$\begin{aligned} p_{\alpha\beta} &= 2E\mathcal{H}_{\alpha\beta} \; , \\ q_{\alpha\beta} &= (2E)^2\, \mathcal{H}^{-1} \det \mathcal{H} \end{aligned} \tag{8}$$

in vacuum and their effective counterparts

$$\begin{aligned} \tilde{p}_{\alpha\beta} &= 2E\tilde{\mathcal{H}}_{\alpha\beta} \; , \\ \tilde{q}_{\alpha\beta} &= (2E)^2\, \tilde{\mathcal{H}}^{-1} \det \tilde{\mathcal{H}} \end{aligned} \tag{9}$$

in matter [14], where the subscripts α and β run over e, μ and τ. The determinants of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ in (8) and (9) are simply $\det\mathcal{H} = m_1^2 m_2^2 m_3^2/(2E)^3$ and $\det\tilde{\mathcal{H}} = \tilde{m}_1^2\tilde{m}_2^2\tilde{m}_3^2/(2E)^3$. In terms of m_i^2 (or $\tilde{m}_i^2$) and $V_{\alpha i}V_{\beta i}^*$ (or $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$), one can obtain

$$\begin{aligned} p_{\alpha\beta} &= \sum_{i=1}^{3}\left(m_i^2 V_{\alpha i}V_{\beta i}^*\right) \; , \\ \tilde{p}_{\alpha\beta} &= \sum_{i=1}^{3}\left(\tilde{m}_i^2 \tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*\right) \; ; \end{aligned} \tag{10}$$

and

$$\begin{aligned} q_{\alpha\beta} &= \frac{1}{2}\sum_{k=1}^{3}\left(m_i^2 m_j^2 V_{\alpha k}V_{\beta k}^*\epsilon_{ijk}^2\right) \; , \\ \tilde{q}_{\alpha\beta} &= \frac{1}{2}\sum_{k=1}^{3}\left(\tilde{m}_i^2\tilde{m}_j^2 \tilde{V}_{\alpha k}\tilde{V}_{\beta k}^*\epsilon_{ijk}^2\right) \; . \end{aligned} \tag{11}$$

Although the exact analytical relations between $\tilde{V}_{\alpha i}$ and $V_{\alpha i}$ have been derived in [11], they are not simple enough to calculate $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ (or $\tilde{V}_{\alpha i}\tilde{V}_{\alpha j}^*$), which are directly relevant to the leptonic unitarity triangles in matter. Hence we shall establish the relations between $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ and $V_{\alpha i}V_{\beta i}^*$ in a different and simpler way. Our strategy is as follows: first, we express $V_{\alpha i}V_{\beta i}^*$ in terms of $(p_{\alpha\beta}, q_{\alpha\beta})$ and $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ in terms of $(\tilde{p}_{\alpha\beta}, \tilde{q}_{\alpha\beta})$; second, we find out the relationship between $(p_{\alpha\beta}, q_{\alpha\beta})$ and $(\tilde{p}_{\alpha\beta}, \tilde{q}_{\alpha\beta})$; finally, we derive the direct relations between $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ and $V_{\alpha i}V_{\beta i}^*$.

Equations (3), (5), (10) and (11) allow us to express $V_{\alpha i}V_{\beta i}^*$ in terms of $(p_{\alpha\beta}, q_{\alpha\beta})$ and $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ in terms of $(\tilde{p}_{\alpha\beta}, \tilde{q}_{\alpha\beta})$. To see this point more clearly, we introduce the coefficient matrices O and $\tilde{O}$:

$$\begin{aligned} O &= \begin{pmatrix} 1 & 1 & 1\\ m_1^2 & m_2^2 & m_3^2\\ m_2^2 m_3^2 & m_1^2 m_3^2 & m_1^2 m_2^2\end{pmatrix} \; , \\ \tilde{O} &= \begin{pmatrix} 1 & 1 & 1\\ \tilde{m}_1^2 & \tilde{m}_2^2 & \tilde{m}_3^2\\ \tilde{m}_2^2 \tilde{m}_3^2 & \tilde{m}_1^2 \tilde{m}_3^2 & \tilde{m}_1^2 \tilde{m}_2^2\end{pmatrix} \; . \end{aligned} \tag{12}$$

Because the three neutrino masses are not exactly degenerate, the inverse matrices of O and $\tilde{O}$ exist. Then one can obtain

$$\begin{aligned} \begin{pmatrix} V_{\alpha 1}V_{\beta 1}^*\\ V_{\alpha 2}V_{\beta 2}^*\\ V_{\alpha 3}V_{\beta 3}^*\end{pmatrix} &= O^{-1}\begin{pmatrix}\delta_{\alpha\beta}\\ p_{\alpha\beta}\\ q_{\alpha\beta}\end{pmatrix} \; , \\ \begin{pmatrix} \tilde{V}_{\alpha 1}\tilde{V}_{\beta 1}^*\\ \tilde{V}_{\alpha 2}\tilde{V}_{\beta 2}^*\\ \tilde{V}_{\alpha 3}\tilde{V}_{\beta 3}^*\end{pmatrix} &= \tilde{O}^{-1}\begin{pmatrix}\delta_{\alpha\beta}\\ \tilde{p}_{\alpha\beta}\\ \tilde{q}_{\alpha\beta}\end{pmatrix} \; . \end{aligned} \tag{13}$$

Taking account of (6) and (7), we find that $(p_{\alpha\beta}, q_{\alpha\beta})$ and $(\tilde{p}_{\alpha\beta}, \tilde{q}_{\alpha\beta})$ are connected with each other through

$$\begin{aligned} \tilde{p}_{\alpha\beta} &= p_{\alpha\beta} + A\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \; , \\ \tilde{q}_{\alpha\beta} &= q_{\alpha\beta} + A\begin{pmatrix}0 & 0 & 0\\ 0 & p_{\tau\tau} & -p_{\tau\mu}\\ 0 & -p_{\mu\tau} & p_{\mu\mu}\end{pmatrix} \; , \end{aligned} \tag{14}$$

where $A \equiv 2Ea$. A combination of (13) and (14) will lead to the direct relations between $\tilde{V}_{\alpha i}\tilde{V}_{\beta i}^*$ and $V_{\alpha i}V_{\beta i}^*$, as one can see in the subsequent sections. The usefulness of (10)–(14) for discussing the matter-induced properties of lepton flavor mixing and CP violation has partly shown up in the literature (see, e.g., [7,14–16]) and will be further demonstrated in the following.

It is worth mentioning that the generic results obtained above are only valid for neutrinos propagating in vacuum and in matter. As for antineutrinos, the corresponding formulas can straightforwardly be written out from (6)–(14) through the replacements $V \Longrightarrow V^*$ and $A \Longrightarrow -A$.

3 Moduli ($\alpha = \beta$)

We first derive the relations between $|\tilde{V}_{\alpha i}|^2$ and $|V_{\alpha i}|^2$ by taking $\alpha = \beta$ for (13). Once (14) is taken into account, we may concretely obtain

$$\begin{pmatrix}|\tilde{V}_{e1}|^2\\|\tilde{V}_{e2}|^2\\|\tilde{V}_{e3}|^2\end{pmatrix} = \tilde{O}^{-1}O\begin{pmatrix}|V_{e1}|^2\\|V_{e2}|^2\\|V_{e3}|^3\end{pmatrix} + A\tilde{O}^{-1}\begin{pmatrix}0\\1\\0\end{pmatrix} ,$$

$$\begin{pmatrix}|\tilde{V}_{\mu1}|^2\\|\tilde{V}_{\mu2}|^2\\|\tilde{V}_{\mu3}|^2\end{pmatrix} = \tilde{O}^{-1}O\begin{pmatrix}|V_{\mu1}|^2\\|V_{\mu2}|^2\\|V_{\mu3}|^3\end{pmatrix} + A\tilde{O}^{-1}T\begin{pmatrix}|V_{\tau1}|^2\\|V_{\tau2}|^2\\|V_{\tau3}|^2\end{pmatrix} ,$$

$$\begin{pmatrix}|\tilde{V}_{\tau1}|^2\\|\tilde{V}_{\tau2}|^2\\|\tilde{V}_{\tau3}|^2\end{pmatrix} = \tilde{O}^{-1}O\begin{pmatrix}|V_{\tau1}|^2\\|V_{\tau2}|^2\\|V_{\tau3}|^2\end{pmatrix} + A\tilde{O}^{-1}T\begin{pmatrix}|V_{\mu1}|^2\\|V_{\mu2}|^2\\|V_{\mu3}|^3\end{pmatrix} , \quad (15)$$

where T is defined as

$$T = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ m_1^2 & m_2^2 & m_3^2\end{pmatrix} . \quad (16)$$

In (15), the inverse matrix of $\tilde{O}$ reads

$$\tilde{O}^{-1} = \frac{1}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}\widetilde{\Delta}_{31}}\begin{pmatrix}\tilde{m}_1^2\widetilde{\Delta}_{23}(\tilde{m}_2^2+\tilde{m}_3^2) & \tilde{m}_1^2\widetilde{\Delta}_{23} & \widetilde{\Delta}_{23}\\ \tilde{m}_2^2\widetilde{\Delta}_{31}(\tilde{m}_1^2+\tilde{m}_3^2) & \tilde{m}_2^2\widetilde{\Delta}_{31} & \widetilde{\Delta}_{31}\\ \tilde{m}_3^2\widetilde{\Delta}_{12}(\tilde{m}_1^2+\tilde{m}_2^2) & \tilde{m}_3^2\widetilde{\Delta}_{12} & \widetilde{\Delta}_{12}\end{pmatrix} , \quad (17)$$

where $\widetilde{\Delta}_{ij} \equiv \tilde{m}_i^2 - \tilde{m}_j^2$. With the help of (3) and (5) as well as the relationship [7]

$$\sum_{i=1}^{3}\tilde{m}_i^2 = \sum_{i=1}^{3}m_i^2 + A , \quad (18)$$

we solve (15) and arrive at

$$|\tilde{V}_{e1}|^2 = \frac{\Delta_{31}\widehat{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{21}}|V_{e1}|^2 + \frac{\Delta_{32}\widehat{\Delta}_{11}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{e2}|^2 + \frac{\widehat{\Delta}_{11}\widehat{\Delta}_{21}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}} ,$$

$$|\tilde{V}_{e2}|^2 = \frac{\Delta_{12}\widehat{\Delta}_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{32}}|V_{e2}|^2 + \frac{\Delta_{13}\widehat{\Delta}_{22}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{e3}|^2 + \frac{\widehat{\Delta}_{22}\widehat{\Delta}_{32}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}} ,$$

$$|\tilde{V}_{e3}|^2 = \frac{\Delta_{23}\widehat{\Delta}_{13}}{\widetilde{\Delta}_{23}\widetilde{\Delta}_{13}}|V_{e3}|^2 + \frac{\Delta_{21}\widehat{\Delta}_{33}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{e1}|^2 + \frac{\widehat{\Delta}_{33}\widehat{\Delta}_{13}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}} ; \quad (19)$$

and

$$|\tilde{V}_{\mu1}|^2 = \frac{\Delta_{31}\widehat{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{21}}|V_{\mu1}|^2 + \frac{\Delta_{32}\widehat{\Delta}_{11}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\mu2}|^2 + \frac{A\Delta_{13}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\tau1}|^2 + \frac{A\Delta_{23}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\tau2}|^2 + C_1 ,$$

$$|\tilde{V}_{\mu2}|^2 = \frac{\Delta_{12}\widehat{\Delta}_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{32}}|V_{\mu2}|^2 + \frac{\Delta_{13}\widehat{\Delta}_{22}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\mu3}|^2 + \frac{A\Delta_{21}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\tau2}|^2 + \frac{A\Delta_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\tau3}|^2 + C_2 ,$$

$$|\tilde{V}_{\mu3}|^2 = \frac{\Delta_{23}\widehat{\Delta}_{13}}{\widetilde{\Delta}_{23}\widetilde{\Delta}_{13}}|V_{\mu3}|^2 + \frac{\Delta_{21}\widehat{\Delta}_{33}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\mu1}|^2 + \frac{A\Delta_{32}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\tau3}|^2 + \frac{A\Delta_{12}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\tau1}|^2 + C_3 ; \quad (20)$$

and

$$|\tilde{V}_{\tau1}|^2 = \frac{\Delta_{31}\widehat{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{21}}|V_{\tau1}|^2 + \frac{\Delta_{32}\widehat{\Delta}_{11}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\tau2}|^2 + \frac{A\Delta_{13}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\mu1}|^2 + \frac{A\Delta_{23}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}}|V_{\mu2}|^2 + C_1 ,$$

$$|\tilde{V}_{\tau2}|^2 = \frac{\Delta_{12}\widehat{\Delta}_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{32}}|V_{\tau2}|^2 + \frac{\Delta_{13}\widehat{\Delta}_{22}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\tau3}|^2 + \frac{A\Delta_{21}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\mu2}|^2 + \frac{A\Delta_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}}|V_{\mu3}|^2 + C_2 ,$$

$$|\tilde{V}_{\tau3}|^2 = \frac{\Delta_{23}\widehat{\Delta}_{13}}{\widetilde{\Delta}_{23}\widetilde{\Delta}_{13}}|V_{\tau3}|^2 + \frac{\Delta_{21}\widehat{\Delta}_{33}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\tau1}|^2 + \frac{A\Delta_{32}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\mu3}|^2 + \frac{A\Delta_{12}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}|V_{\mu1}|^2 + C_3 , \quad (21)$$

where $\Delta_{ij} \equiv m_i^2 - m_j^2$, $\widehat{\Delta}_{ij} \equiv m_i^2 - \tilde{m}_j^2$, and

$$C_1 = -\frac{\Delta_{31}\Delta_{23} + \widehat{\Delta}_{31}\widehat{\Delta}_{32} + \widehat{\Delta}_{31}\widehat{\Delta}_{33}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}} ,$$

$$C_2 = -\frac{\Delta_{12}\Delta_{31} + \widehat{\Delta}_{12}\widehat{\Delta}_{13} + \widehat{\Delta}_{12}\widehat{\Delta}_{11}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{23}} ,$$

$$C_3 = -\frac{\Delta_{23}\Delta_{12} + \widehat{\Delta}_{23}\widehat{\Delta}_{21} + \widehat{\Delta}_{23}\widehat{\Delta}_{22}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}} . \quad (22)$$

In Appendix A, we list the explicit expressions of $\widetilde{\Delta}_{ij}$ and $\widehat{\Delta}_{ij}$ in terms of Δ_{ij} and A. It is then possible to evaluate the deviation of $|\tilde{V}_{\alpha i}|^2$ from $|V_{\alpha i}|^2$ by using the formulas obtained above. Of course, $|\tilde{V}_{\alpha i}|^2 = |V_{\alpha i}|^2$ holds at the limit $A \to 0$ or $E \to 0$.

Two sets of normalization conditions given in (5) allow us to define two off-diagonal asymmetries of $\tilde{V}$:

$$\tilde{\mathcal{A}}_{\rm L} \equiv |\tilde{V}_{e2}|^2 - |\tilde{V}_{\mu1}|^2 = |\tilde{V}_{\mu3}|^2 - |\tilde{V}_{\tau2}|^2 = |\tilde{V}_{\tau1}|^2 - |\tilde{V}_{e3}|^2 ,$$
$$\tilde{\mathcal{A}}_{\rm R} \equiv |\tilde{V}_{e2}|^2 - |\tilde{V}_{\mu3}|^2 = |\tilde{V}_{\mu1}|^2 - |\tilde{V}_{\tau2}|^2 = |\tilde{V}_{\tau3}|^2 - |\tilde{V}_{e1}|^2 . \quad (23)$$

They can straightforwardly be calculated by using (19), (20) and (21). Comparing between $\tilde{\mathcal{A}}_{\rm L}$ (or $\tilde{\mathcal{A}}_{\rm R}$) and its counterpart $\mathcal{A}_{\rm L}$ (or $\mathcal{A}_{\rm R}$) in vacuum, we may examine the matter effect on the geometric structure of V. In particular, the six matter-modified unitarity triangles will reduce to three pairs of *congruent* triangles [17], if $\tilde{\mathcal{A}}_{\rm L} = 0$ or $\tilde{\mathcal{A}}_{\rm R} = 0$ holds.

The sides of six unitarity triangles can also be obtained from (19), (20) and (21). If the three sides of a specific triangle are comparable in magnitude, one may use them to calculate the area of this triangle – namely, the CP violating invariant $\tilde{\mathcal{J}}$. Indeed, $\tilde{\mathcal{J}}$ is given by

$$\tilde{\mathcal{J}}^2 = |\tilde{V}_{\alpha i}|^2|\tilde{V}_{\beta j}|^2|\tilde{V}_{\alpha j}|^2|\tilde{V}_{\beta i}|^2 - \frac{1}{4}\Big(1 + |\tilde{V}_{\alpha i}|^2|\tilde{V}_{\beta j}|^2 + |\tilde{V}_{\alpha j}|^2|\tilde{V}_{\beta i}|^2 \quad (24)$$

$$-|\tilde{V}_{\alpha i}|^2 - |\tilde{V}_{\beta j}|^2 - |\tilde{V}_{\alpha j}|^2 - |\tilde{V}_{\beta i}|^2\Big)^2 \ ,$$

in which $\alpha \neq \beta$ running over (e, μ, τ) and $i \neq j$ running over $(1, 2, 3)$. The implication of this result is obvious: important information about leptonic CP violation can in principle be extracted from the measured moduli of four independent matrix elements of $\tilde{V}$.

4 Angles ($\alpha \neq \beta$)

Now let us focus our attention on the inner angles of six unitarity triangles. There are only nine independent angles, which appear either in triangles $\triangle_{e,\mu,\tau}$ or in triangles $\triangle_{1,2,3}$ (see Fig. 1 for illustration). Hence it is only necessary to consider $(\triangle_e, \triangle_\mu, \triangle_\tau)$ and their effective counterparts $(\tilde{\triangle}_e, \tilde{\triangle}_\mu, \tilde{\triangle}_\tau)$ in the calculation of nine angles. To be specific, an inner angle of Δ_α or $\tilde{\Delta}_\alpha$ (for $\alpha = e, \mu$ or τ) can be defined as

$$\begin{aligned} \phi^{ij}_{\alpha\beta} &\equiv \arg\left(-\frac{V_{\alpha i} V^*_{\beta i}}{V_{\alpha j} V^*_{\beta j}}\right) \ , \\ \tilde{\phi}^{ij}_{\alpha\beta} &\equiv \arg\left(-\frac{\tilde{V}_{\alpha i} \tilde{V}^*_{\beta i}}{\tilde{V}_{\alpha j} \tilde{V}^*_{\beta j}}\right) \ , \end{aligned} \tag{25}$$

where (α, β) run over (e, μ), (μ, τ) and (τ, e), and (i, j) run over $(1, 2)$, $(2, 3)$ and $(3, 1)$. Taking account of (2) and (4), we obtain

$$\begin{aligned} \cot\phi^{ij}_{\alpha\beta} &= \frac{1}{\mathcal{J}} \mathrm{Re}\left(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}\right) \ , \\ \cot\tilde{\phi}^{ij}_{\alpha\beta} &= \frac{1}{\tilde{\mathcal{J}}} \mathrm{Re}\left(\tilde{V}_{\alpha i} \tilde{V}_{\beta j} \tilde{V}^*_{\alpha j} \tilde{V}^*_{\beta i}\right) \ . \end{aligned} \tag{26}$$

Note that $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are related with each other through the equation [18]

$$\tilde{\mathcal{J}} \widetilde{\Delta}_{12} \widetilde{\Delta}_{13} \widetilde{\Delta}_{23} \ = \ \mathcal{J} \Delta_{12} \Delta_{13} \Delta_{23} \ . \tag{27}$$

In order to link $\cot\tilde{\phi}^{ij}_{\alpha\beta}$ to $\cot\phi^{ij}_{\alpha\beta}$ via (26), we need to find out the relationship between $\mathrm{Re}(\tilde{V}_{\alpha i} \tilde{V}_{\beta j} \tilde{V}^*_{\alpha j} \tilde{V}^*_{\beta i})$ and $\mathrm{Re}(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i})$.

For $\alpha \neq \beta$, (13) and (14) yield

$$\begin{aligned} \begin{pmatrix} \tilde{V}_{e1}\tilde{V}^*_{\mu 1} \\ \tilde{V}_{e2}\tilde{V}^*_{\mu 2} \\ \tilde{V}_{e3}\tilde{V}^*_{\mu 3} \end{pmatrix} &= \tilde{O}^{-1} O \begin{pmatrix} V_{e1}V^*_{\mu 1} \\ V_{e2}V^*_{\mu 2} \\ V_{e3}V^*_{\mu 3} \end{pmatrix} \ , \\ \begin{pmatrix} \tilde{V}_{e1}\tilde{V}^*_{\tau 1} \\ \tilde{V}_{e2}\tilde{V}^*_{\tau 2} \\ \tilde{V}_{e3}\tilde{V}^*_{\tau 3} \end{pmatrix} &= \tilde{O}^{-1} O \begin{pmatrix} V_{e1}V^*_{\tau 1} \\ V_{e2}V^*_{\tau 2} \\ V_{e3}V^*_{\tau 3} \end{pmatrix} \ , \\ \begin{pmatrix} \tilde{V}_{\mu 1}\tilde{V}^*_{\tau 1} \\ \tilde{V}_{\mu 2}\tilde{V}^*_{\tau 2} \\ \tilde{V}_{\mu 3}\tilde{V}^*_{\tau 3} \end{pmatrix} &= \tilde{O}^{-1} O \begin{pmatrix} V_{\mu 1}V^*_{\tau 1} \\ V_{\mu 2}V^*_{\tau 2} \\ V_{\mu 3}V^*_{\tau 3} \end{pmatrix} - A\tilde{O}^{-1} T \begin{pmatrix} V_{\mu 1}V^*_{\tau 1} \\ V_{\mu 2}V^*_{\tau 2} \\ V_{\mu 3}V^*_{\tau 3} \end{pmatrix} \ , \end{aligned} \tag{28}$$

where T has been defined in (16). With the help of (17) and (18), one may solve (28) and obtain the matter-induced corrections to $V_{\alpha i} V^*_{\beta i}$. Three sides of the effective unitarity triangle $\tilde{\triangle}_e$, $\tilde{\triangle}_\mu$ or $\tilde{\triangle}_\tau$ are then given by

$$\begin{aligned} \tilde{V}_{e1}\tilde{V}^*_{\mu 1} &= \frac{\widehat{\Delta}_{21}\Delta_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}} V_{e1}V^*_{\mu 1} + \frac{\widehat{\Delta}_{11}\Delta_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}} V_{e2}V^*_{\mu 2} \ , \\ \tilde{V}_{e2}\tilde{V}^*_{\mu 2} &= \frac{\widehat{\Delta}_{32}\Delta_{21}}{\widetilde{\Delta}_{32}\widetilde{\Delta}_{21}} V_{e2}V^*_{\mu 2} + \frac{\widehat{\Delta}_{22}\Delta_{31}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}} V_{e3}V^*_{\mu 3} \ , \\ \tilde{V}_{e3}\tilde{V}^*_{\mu 3} &= \frac{\widehat{\Delta}_{13}\Delta_{23}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{e3}V^*_{\mu 3} + \frac{\widehat{\Delta}_{33}\Delta_{21}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{e1}V^*_{\mu 1} \end{aligned} \tag{29}$$

for $\tilde{\triangle}_\tau$; and

$$\begin{aligned} \tilde{V}_{\tau 1}\tilde{V}^*_{e1} &= \frac{\widehat{\Delta}_{21}\Delta_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}} V_{\tau 1}V^*_{e1} + \frac{\widehat{\Delta}_{11}\Delta_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}} V_{\tau 2}V^*_{e2} \ , \\ \tilde{V}_{\tau 2}\tilde{V}^*_{e2} &= \frac{\widehat{\Delta}_{32}\Delta_{21}}{\widetilde{\Delta}_{32}\widetilde{\Delta}_{21}} V_{\tau 2}V^*_{e2} + \frac{\widehat{\Delta}_{22}\Delta_{31}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}} V_{\tau 3}V^*_{e3} \ , \\ \tilde{V}_{\tau 3}\tilde{V}^*_{e3} &= \frac{\widehat{\Delta}_{13}\Delta_{23}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{\tau 3}V^*_{e3} + \frac{\widehat{\Delta}_{33}\Delta_{21}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{\tau 1}V^*_{e1} \end{aligned} \tag{30}$$

for $\tilde{\triangle}_\mu$; and

$$\begin{aligned} \tilde{V}_{\mu 1}\tilde{V}^*_{\tau 1} &= \frac{(\widehat{\Delta}_{21} + A)\Delta_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}} V_{\mu 1}V^*_{\tau 1} + \frac{(\widehat{\Delta}_{11} + A)\Delta_{32}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{13}} V_{\mu 2}V^*_{\tau 2} \ , \\ \tilde{V}_{\mu 2}\tilde{V}^*_{\tau 2} &= \frac{(\widehat{\Delta}_{32} + A)\Delta_{21}}{\widetilde{\Delta}_{32}\widetilde{\Delta}_{21}} V_{\mu 2}V^*_{\tau 2} + \frac{(\widehat{\Delta}_{22} + A)\Delta_{31}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}} V_{\mu 3}V^*_{\tau 3} \ , \\ \tilde{V}_{\mu 3}\tilde{V}^*_{\tau 3} &= \frac{(\widehat{\Delta}_{13} + A)\Delta_{23}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{\mu 3}V^*_{\tau 3} + \frac{(\widehat{\Delta}_{33} + A)\Delta_{21}}{\widetilde{\Delta}_{13}\widetilde{\Delta}_{23}} V_{\mu 1}V^*_{\tau 1} \end{aligned} \tag{31}$$

for $\tilde{\triangle}_e$. It is remarkable that (29), (30) or (31), together with (2) and (4), can be used to derive (27). The relationship between $\mathrm{Re}(\tilde{V}_{\alpha i} \tilde{V}_{\beta j} \tilde{V}^*_{\alpha j} \tilde{V}^*_{\beta i})$ and $\mathrm{Re}(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i})$ can also be derived from these equations. Then we are able to establish the direct connection between $\cot\tilde{\phi}^{ij}_{\alpha\beta}$ and $\cot\phi^{ij}_{\alpha\beta}$.

Comparing between the definition of $\phi^{ij}_{\alpha\beta}$ and the simpler notation of nine inner angles in Fig. 1, we have $\angle 1 = \phi^{12}_{\mu\tau}$, $\angle 2 = \phi^{23}_{\mu\tau}$, $\angle 3 = \phi^{31}_{\mu\tau}$; $\angle 4 = \phi^{12}_{\tau e}$, $\angle 5 = \phi^{23}_{\tau e}$, $\angle 6 = \phi^{31}_{\tau e}$; $\angle 7 = \phi^{12}_{e\mu}$, $\angle 8 = \phi^{23}_{e\mu}$, $\angle 9 = \phi^{31}_{e\mu}$. One may use the similar notation ($\widetilde{\angle}1$, $\cdots$, $\widetilde{\angle}9$) to replace $\tilde{\phi}^{ij}_{\alpha\beta}$ for the matter-modified unitarity triangles. Therefore,

$$\begin{aligned} \cot\widetilde{\angle}1 = & \ \frac{(\widehat{\Delta}_{32} + A)(\widehat{\Delta}_{21} + A)}{\Delta_{32}\widetilde{\Delta}_{21}} \cot\angle 1 \\ & + \frac{(\widehat{\Delta}_{11} + A)(\widehat{\Delta}_{22} + A)}{\Delta_{12}\widetilde{\Delta}_{12}} \cot\angle 2 \\ & + \frac{(\widehat{\Delta}_{21} + A)(\widehat{\Delta}_{22} + A)\Delta_{13}}{\Delta_{12}\Delta_{23}\widetilde{\Delta}_{12}} \cot\angle 3 \\ & + \frac{(\widehat{\Delta}_{11} + A)(\widehat{\Delta}_{32} + A)}{\mathcal{J}\Delta_{31}\widetilde{\Delta}_{12}} |V_{\mu 2}V^*_{\tau 2}|^2 \ , \end{aligned}$$

Table 1. Numerical illustration of the terrestrial matter effect on nine inner angles of six effective unitarity triangles ($\tilde{\triangle}_{e,\mu,\tau}$ and $\tilde{\triangle}_{1,2,3}$) in a long-baseline neutrino oscillation experiment, where ν stands for neutrinos and $\overline{\nu}$ represents antineutrinos

Angle	$\tilde{V} = V$	$E = 1\,\text{GeV}$	$E = 2\,\text{GeV}$	$E = 3\,\text{GeV}$
$\widetilde{\angle}1$	166.9°	143.3° (ν)	104.9° (ν)	73.4° (ν)
		141.2° ($\overline{\nu}$)	119.0° ($\overline{\nu}$)	103.2° ($\overline{\nu}$)
$\widetilde{\angle}2$	3.9°	35.5° (ν)	74.5° (ν)	106.1° (ν)
		0.8° ($\overline{\nu}$)	0.4° ($\overline{\nu}$)	0.2° ($\overline{\nu}$)
$\widetilde{\angle}3$	9.2°	1.2° (ν)	0.6° (ν)	0.5° (ν)
		38.1° ($\overline{\nu}$)	60.7° ($\overline{\nu}$)	76.6° ($\overline{\nu}$)
$\widetilde{\angle}4$	6.6°	18.4° (ν)	37.6° (ν)	53.3° (ν)
		19.4° ($\overline{\nu}$)	30.5° ($\overline{\nu}$)	38.4° ($\overline{\nu}$)
$\widetilde{\angle}5$	88.1°	72.2° (ν)	52.7° (ν)	36.9° (ν)
		89.6° ($\overline{\nu}$)	89.8° ($\overline{\nu}$)	89.9° ($\overline{\nu}$)
$\widetilde{\angle}6$	85.4°	89.4° (ν)	89.7° (ν)	89.8° (ν)
		71.0° ($\overline{\nu}$)	59.7° ($\overline{\nu}$)	51.7° ($\overline{\nu}$)
$\widetilde{\angle}7$	6.6°	18.4° (ν)	37.6° (ν)	53.3° (ν)
		19.4° ($\overline{\nu}$)	30.5° ($\overline{\nu}$)	38.4° ($\overline{\nu}$)
$\widetilde{\angle}8$	88.1°	72.2° (ν)	52.7° (ν)	36.9° (ν)
		89.6° ($\overline{\nu}$)	89.8° ($\overline{\nu}$)	89.9° ($\overline{\nu}$)
$\widetilde{\angle}9$	85.4°	89.4° (ν)	89.7° (ν)	89.8° (ν)
		71.0° ($\overline{\nu}$)	59.7° ($\overline{\nu}$)	51.7° ($\overline{\nu}$)

$$
\begin{aligned}
\cot\widetilde{\angle}2 &= \frac{(\hat{\Delta}_{13}+A)(\hat{\Delta}_{32}+A)}{\Delta_{13}\tilde{\Delta}_{32}}\cot\angle 2 \\
&+ \frac{(\hat{\Delta}_{22}+A)(\hat{\Delta}_{33}+A)}{\Delta_{23}\tilde{\Delta}_{23}}\cot\angle 3 \\
&+ \frac{(\hat{\Delta}_{32}+A)(\hat{\Delta}_{33}+A)\Delta_{21}}{\Delta_{23}\Delta_{31}\tilde{\Delta}_{23}}\cot\angle 1 \\
&+ \frac{(\hat{\Delta}_{22}+A)(\hat{\Delta}_{13}+A)}{\mathcal{J}\Delta_{12}\tilde{\Delta}_{23}}|V_{\mu 3}V^*_{\tau 3}|^2 \; , \\
\cot\widetilde{\angle}3 &= \frac{(\hat{\Delta}_{21}+A)(\hat{\Delta}_{13}+A)}{\Delta_{21}\tilde{\Delta}_{13}}\cot\angle 3 \\
&+ \frac{(\hat{\Delta}_{11}+A)(\hat{\Delta}_{33}+A)}{\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 1 \\
&+ \frac{(\hat{\Delta}_{11}+A)(\hat{\Delta}_{13}+A)\Delta_{32}}{\Delta_{12}\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 2 \\
&+ \frac{(\hat{\Delta}_{21}+A)(\hat{\Delta}_{33}+A)}{\mathcal{J}\Delta_{23}\tilde{\Delta}_{31}}|V_{\mu 1}V^*_{\tau 1}|^2 \; ; \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
\cot\widetilde{\angle}4 &= \frac{\hat{\Delta}_{32}\hat{\Delta}_{21}}{\Delta_{32}\tilde{\Delta}_{21}}\cot\angle 4 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{22}}{\Delta_{12}\tilde{\Delta}_{12}}\cot\angle 5 \\
&+ \frac{\hat{\Delta}_{21}\hat{\Delta}_{22}\Delta_{13}}{\Delta_{12}\Delta_{23}\tilde{\Delta}_{12}}\cot\angle 6 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{32}}{\mathcal{J}\Delta_{31}\tilde{\Delta}_{12}}|V_{\tau 2}V^*_{e2}|^2 \; , \\
\cot\widetilde{\angle}5 &= \frac{\hat{\Delta}_{13}\hat{\Delta}_{32}}{\Delta_{13}\tilde{\Delta}_{32}}\cot\angle 5 + \frac{\hat{\Delta}_{22}\hat{\Delta}_{33}}{\Delta_{23}\tilde{\Delta}_{23}}\cot\angle 6 \\
&+ \frac{\hat{\Delta}_{32}\hat{\Delta}_{33}\Delta_{21}}{\Delta_{23}\Delta_{31}\tilde{\Delta}_{23}}\cot\angle 4 + \frac{\hat{\Delta}_{22}\hat{\Delta}_{13}}{\mathcal{J}\Delta_{12}\tilde{\Delta}_{23}}|V_{\tau 3}V^*_{e3}|^2 \; , \\
\cot\widetilde{\angle}6 &= \frac{\hat{\Delta}_{21}\hat{\Delta}_{13}}{\Delta_{21}\tilde{\Delta}_{13}}\cot\angle 6 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{33}}{\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 4 \qquad (33) \\
&+ \frac{\hat{\Delta}_{11}\hat{\Delta}_{13}\Delta_{32}}{\Delta_{12}\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 5 + \frac{\hat{\Delta}_{21}\hat{\Delta}_{33}}{\mathcal{J}\Delta_{23}\tilde{\Delta}_{31}}|V_{\tau 1}V^*_{e1}|^2 \; ;
\end{aligned}
$$

and

$$
\begin{aligned}
\cot\widetilde{\angle}7 &= \frac{\hat{\Delta}_{32}\hat{\Delta}_{21}}{\Delta_{32}\tilde{\Delta}_{21}}\cot\angle 7 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{22}}{\Delta_{12}\tilde{\Delta}_{12}}\cot\angle 8 \\
&+ \frac{\hat{\Delta}_{21}\hat{\Delta}_{22}\Delta_{13}}{\Delta_{12}\Delta_{23}\tilde{\Delta}_{12}}\cot\angle 9 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{32}}{\mathcal{J}\Delta_{31}\tilde{\Delta}_{12}}|V_{e2}V^*_{\mu 2}|^2 \; , \\
\cot\widetilde{\angle}8 &= \frac{\hat{\Delta}_{13}\hat{\Delta}_{32}}{\Delta_{13}\tilde{\Delta}_{32}}\cot\angle 8 + \frac{\hat{\Delta}_{22}\hat{\Delta}_{33}}{\Delta_{23}\tilde{\Delta}_{23}}\cot\angle 9 \\
&+ \frac{\hat{\Delta}_{32}\hat{\Delta}_{33}\Delta_{21}}{\Delta_{23}\Delta_{31}\tilde{\Delta}_{23}}\cot\angle 7 + \frac{\hat{\Delta}_{22}\hat{\Delta}_{13}}{\mathcal{J}\Delta_{12}\tilde{\Delta}_{23}}|V_{e3}V^*_{\mu 3}|^2 \; , \\
\cot\widetilde{\angle}9 &= \frac{\hat{\Delta}_{21}\hat{\Delta}_{13}}{\Delta_{21}\tilde{\Delta}_{13}}\cot\angle 9 + \frac{\hat{\Delta}_{11}\hat{\Delta}_{33}}{\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 7 \qquad (34) \\
&+ \frac{\hat{\Delta}_{11}\hat{\Delta}_{13}\Delta_{32}}{\Delta_{12}\Delta_{31}\tilde{\Delta}_{31}}\cot\angle 8 + \frac{\hat{\Delta}_{21}\hat{\Delta}_{33}}{\mathcal{J}\Delta_{23}\tilde{\Delta}_{31}}|V_{e1}V^*_{\mu 1}|^2 \; .
\end{aligned}
$$

These exact analytical results clearly show how nine inner angles of six unitarity triangles get modified by the matter effects.

5 Illustration

We proceed to numerically illustrate the matter effects on the shapes of six unitarity triangles, the rephasing invariant of CP violation, and the off-diagonal asymmetries of V. In view of current solar [1] and atmospheric [2] neutrino oscillation data, we typically take $\Delta_{21} \approx 8\times 10^{-5}\,\text{eV}^2$ and $\Delta_{32} \approx 2.3\times 10^{-3}\,\text{eV}^2$. We also take $\theta_{12} \approx 33°$, $\theta_{23} \approx 45°$, $\theta_{13} \approx 3°$ and $\delta \approx 90°$ in the standard parametrization of V [7]. It is unnecessary to specify two Majorana-type CP violating phases of V in our calculations, because they play no role in the configuration of leptonic unitarity triangles [19]. The inputs taken above lead to $\mathcal{J} \approx 0.012$, $\mathcal{A}_{\rm L} \approx 0.147$ and $\mathcal{A}_{\rm R} \approx -0.203$. This means that V is asymmetric both about its V_{e1}–$V_{\mu 2}$–$V_{\tau 3}$ axis and about its V_{e3}–$V_{\mu 2}$–$V_{\tau 1}$ axis, and the area of each triangle is about 8 times smaller than its maximal limit (i.e., $\mathcal{J} = 1/(6\sqrt{3})$ [6]).

For a realistic long-baseline neutrino oscillation experiment, the dependence of terrestrial matter effects on the neutrino beam energy can approximately be written as $A \approx 2.28\times 10^{-4}\,\text{eV}^2 E/[\text{GeV}]$ [13]. This approximation is reasonably good and close to reality only if the baseline length is about 1000 km or shorter [20]. Of course, $\tilde{V} = V$ holds in the limit of $E = 0$ or $A = 0$. Typically taking $E = 1\,\text{GeV}$, $2\,\text{GeV}$ and $3\,\text{GeV}$, we calculate the sides of six effective unitarity triangles in matter and show the changes of their shapes in Figs. 2–5. The corresponding

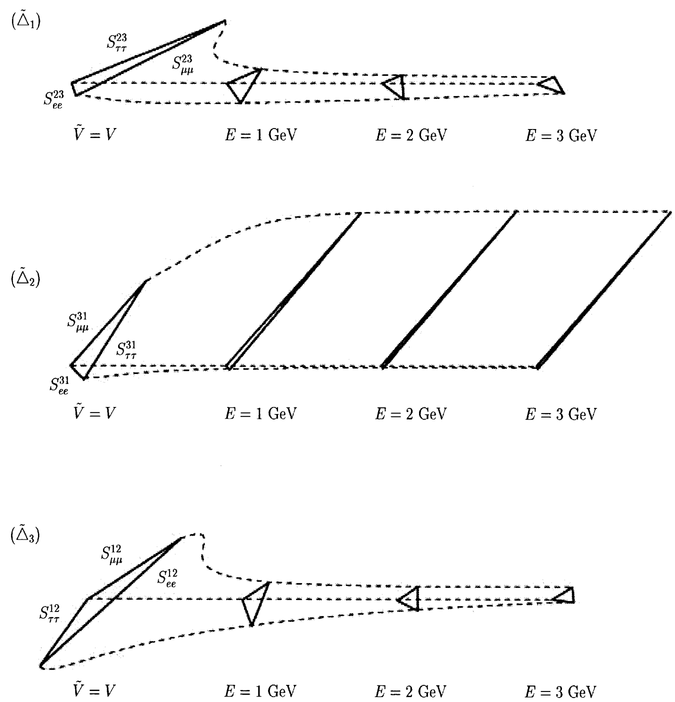

Fig. 2. The shape evolution of three effective unitarity triangles $(\tilde{\triangle}_1, \tilde{\triangle}_2, \tilde{\triangle}_3)$ with the beam energy of neutrinos $(+A$ and $V)$ in a realistic long-baseline oscillation experiment, where $S^{ij}_{\alpha\alpha} \equiv |\tilde{V}_{\alpha i}\tilde{V}^*_{\alpha j}|$ (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) has been defined

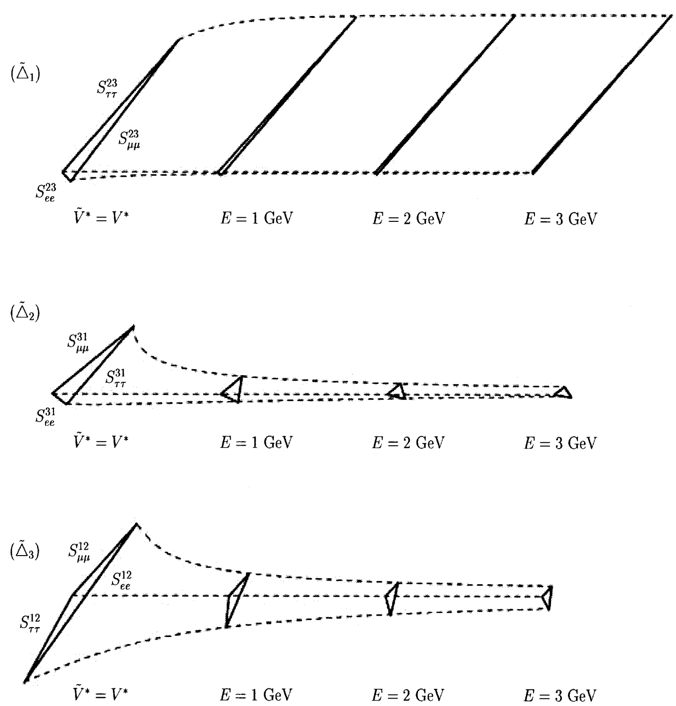

Fig. 3. The shape evolution of three effective unitarity triangles $(\tilde{\triangle}_1, \tilde{\triangle}_2, \tilde{\triangle}_3)$ with the beam energy of antineutrinos $(-A$ and $V^*)$ in a realistic long-baseline oscillation experiment, where $S^{ij}_{\alpha\alpha} \equiv |\tilde{V}_{\alpha i}\tilde{V}^*_{\alpha j}|$ (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) has been defined

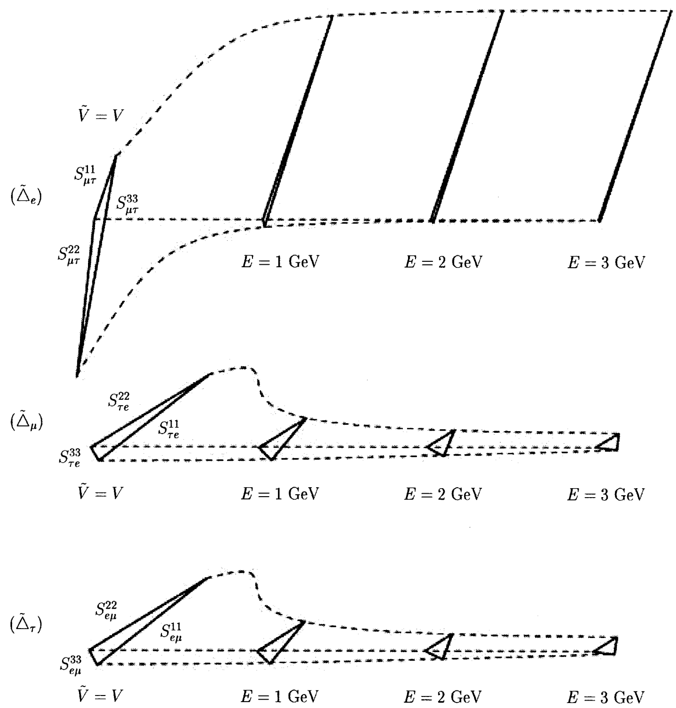

Fig. 4. The shape evolution of three effective unitarity triangles $(\tilde{\triangle}_e, \tilde{\triangle}_\mu, \tilde{\triangle}_\tau)$ with the beam energy of neutrinos $(+A$ and $V)$ in a realistic long-baseline oscillation experiment, where $S^{ii}_{\alpha\beta} \equiv |\tilde{V}_{\alpha i}\tilde{V}^*_{\beta i}|$ (for $i = 1, 2, 3$ and $\alpha, \beta = e, \mu, \tau$) has been defined

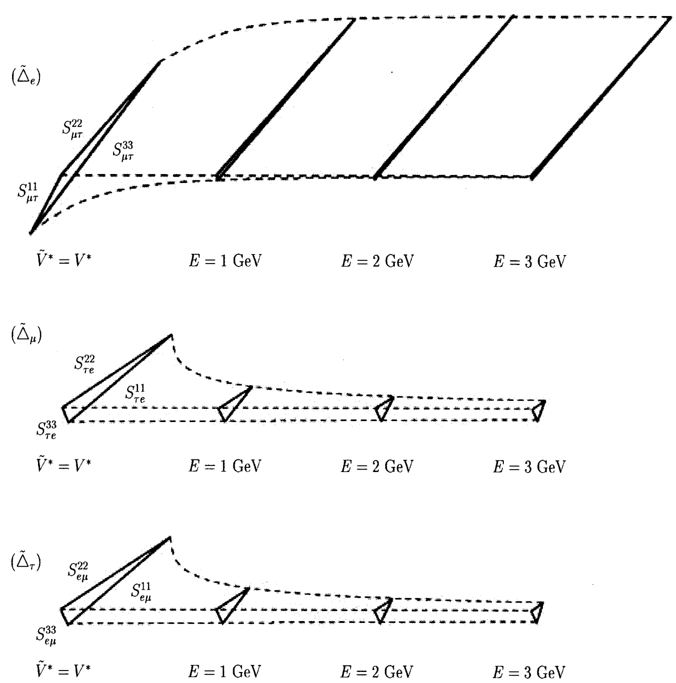

Fig. 5. The shape evolution of three effective unitarity triangles $(\tilde{\triangle}_e, \tilde{\triangle}_\mu, \tilde{\triangle}_\tau)$ with the beam energy of antineutrinos $(-A$ and $V^*)$ in a realistic long-baseline oscillation experiment, where $S^{ii}_{\alpha\beta} \equiv |\tilde{V}_{\alpha i}\tilde{V}^*_{\beta i}|$ (for $i = 1, 2, 3$ and $\alpha, \beta = e, \mu, \tau$) has been defined

results for nine inner angles of $\tilde{\triangle}_{e,\mu,\tau}$ or $\tilde{\triangle}_{1,2,3}$ are presented in Table 1, and the numerical dependence of $\tilde{\mathcal{J}}$, $\tilde{\mathcal{A}}_{\rm L}$ and $\tilde{\mathcal{A}}_{\rm R}$ on E is illustrated in Fig. 6. Some comments and discussions are in order.

(a) The smallness of θ_{13} (or $|V_{e3}|$) makes one side of $\triangle_1$ or $\triangle_2$ strongly suppressed. When the terrestrial matter effect with $E \geq 1\,{\rm GeV}$ is taken into account, three sides of $\tilde{\triangle}_1$ and $\tilde{\triangle}_3$ become comparable in magnitude for neutrinos ($+A$ and V); or three sides of $\tilde{\triangle}_2$ and $\tilde{\triangle}_3$ become comparable in magnitude for antineutrinos ($-A$ and V^*). As a consequence of matter corrections, the shortest side of $\triangle_2$ (i.e., $S^{31}_{ee} \equiv |\tilde{V}_{e3}\tilde{V}^*_{e1}|$ at $A = 0$) turns out to be shorter for neutrinos; so does that of $\triangle_1$ (i.e., $S^{23}_{ee} \equiv |\tilde{V}_{e2}\tilde{V}^*_{e3}|$ at $A = 0$) for antineutrinos. We find that the shape of $\tilde{\triangle}_3$ is relatively stable against matter corrections, no matter whether the neutrino beam or the antineutrino beam is concerned.

(b) Similarly because of the smallness of θ_{13} (or $|V_{e3}|$), one side of $\triangle_\mu$ or $\triangle_\tau$ is strongly suppressed. The terrestrial matter effect becomes significant for $E \geq 1\,{\rm GeV}$. In this case, three sides of $\tilde{\triangle}_\mu$ and $\tilde{\triangle}_\tau$ are comparable in magnitude for either neutrinos ($+A$ and V) or antineutrinos ($-A$ and V^*). An interesting feature of $\tilde{\triangle}_e$ is that its side $S^{22}_{\mu\tau} \equiv |\tilde{V}_{\mu 2}\tilde{V}^*_{\tau 2}|$ is dramatically sensitive to matter corrections and becomes very short for neutrinos, while its side $S^{11}_{\mu\tau} \equiv |\tilde{V}_{\mu 1}\tilde{V}^*_{\tau 1}|$ may significantly be suppressed by matter effects for antineutrinos. One can see that the shapes of $\tilde{\triangle}_\mu$ and $\tilde{\triangle}_\tau$ are relatively stable against matter corrections, no matter whether the neutrino beam or the antineutrino beam is concerned.

(c) Note that the input $\theta_{23} \approx 45^\circ$, which is well favored by current atmospheric neutrino oscillation data [2], results in $|V_{\mu i}| \approx |V_{\tau i}|$ (for $i = 1, 2, 3$). Hence triangles $\triangle_\mu$ and $\triangle_\tau$ are congruent with each other; so are their effective counterparts $\tilde{\triangle}_\mu$ and $\tilde{\triangle}_\tau$. This accidental result has actually shown up in Figs. 4 and 5. From the phenomenological point of view, it makes sense to measure θ_{23} accurately and to examine its possible deviation from maximal mixing (i.e., $\theta_{23} = 45^\circ$ exactly), so as to explore the underlying μ–τ flavor symmetry and its geometric manifestation in leptonic unitarity triangles.

(d) Table 1 is helpful for us to understand the terrestrial matter effect on nine inner angles of six unitarity triangles. One can see that $\tilde{\angle}6$ and $\tilde{\angle}9$, which happen to be identical as a consequence of $\theta_{23} \approx 45^\circ$, are relatively stable against matter corrections. In contrast, $\tilde{\angle}2$, $\tilde{\angle}3$ and $\tilde{\angle}4$ are rather sensitive to matter corrections.

(e) No matter how the sides and angles of six unitarity triangles change with the matter effect, their area ($\tilde{\mathcal{J}}/2$) is in most cases smaller than that in vacuum ($\mathcal{J}/2$). This unfortunate feature, as shown in Fig. 6, makes it hard to directly measure leptonic CP or T violation in any realistic long-baseline neutrino oscillations. If the neutrino beam energy is small (e.g., about 1 GeV or smaller), the matter-induced suppression of $\mathcal{J}$ is not significant. In this case, a study of leptonic CP violation and unitarity triangles in a *medium*-baseline neutrino oscillation experiment seems more feasible [21].

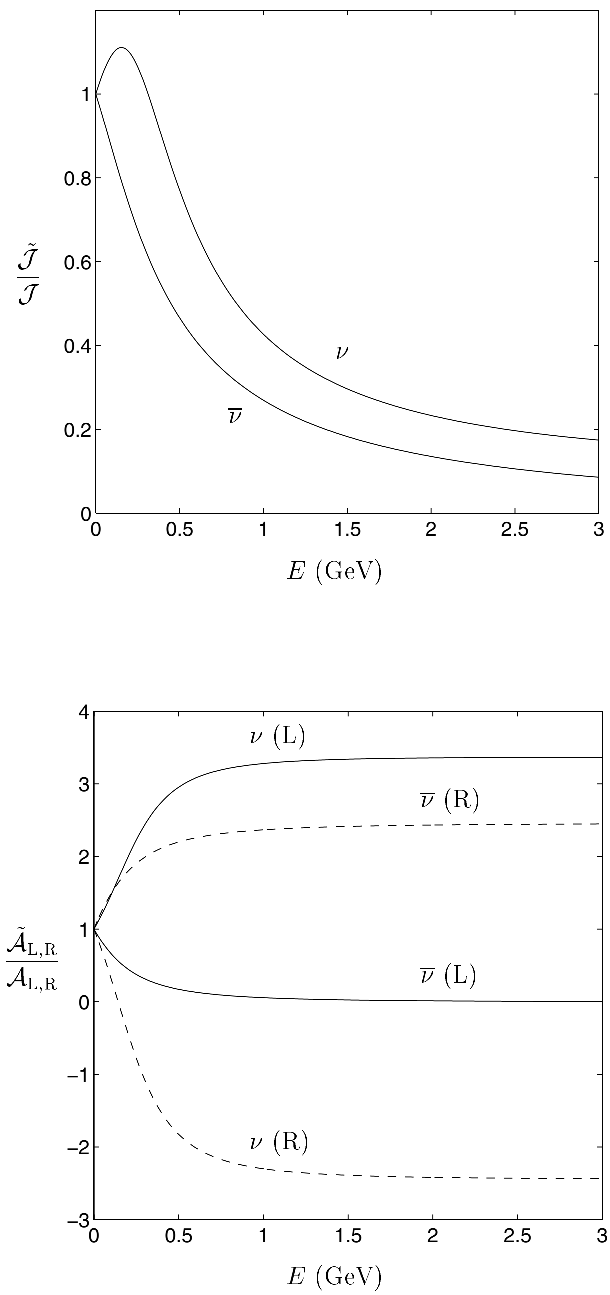

Fig. 6. Terrestrial matter effects on $\mathcal{J}$, $\mathcal{A}_{\rm L}$ and $\mathcal{A}_{\rm R}$ for neutrinos (ν with $+A$ and V) and antineutrinos ($\overline{\nu}$ with $-A$ and V^*) in a realistic long-baseline oscillation experiment

(f) The matter effect on two off-diagonal asymmetries of V is also illustrated in Fig. 6. One can see that $\tilde{\mathcal{A}}_{\rm L} \approx 0$ is a good approximation, when the beam energy of antineutrinos satisfies $E \geq 1\,{\rm GeV}$. In this interesting case, one may approximately arrive at $\tilde{\triangle}_e \cong \tilde{\triangle}_1$, $\tilde{\triangle}_\mu \cong \tilde{\triangle}_2$ and $\tilde{\triangle}_\tau \cong \tilde{\triangle}_3$. Such a result can also be seen from Figs. 3 and 5. The possibility for $\tilde{\mathcal{A}}_{\rm R} \approx 0$ may appear only when the beam energy of neutrinos is around $E \approx 0.2\,{\rm GeV}$, as shown in Fig. 6. These results are certainly dependent upon our inputs. Therefore, they mainly serve for illustration.

In practice, the reconstruction of a unitarity triangle requires a series of measurements, which can be done in both *appearance* and *disappearance* neutrino oscillation experiments. More detailed analyses in this respect can be seen from [9] (see also [22]), where the central attention has only been paid to the triangle $\triangle_\tau$ and its effective counterpart $\tilde{\triangle}_\tau$. A similar analysis of the other triangles is expected to be very lengthy and will be presented elsewhere [23]. As we have emphasized, it is possible to establish leptonic CP violation by means of the precise experimental information about the sides of six unitarity triangles. In this sense, the geometric approach described here may be complimentary to the direct determination of CP violation from the measurement of probability asymmetries between $\nu_\alpha \to \nu_\beta$ and $\overline{\nu}_\alpha \to \overline{\nu}_\beta$ (for $\alpha \neq \beta$ running over e, μ, τ) oscillations.

6 Summary

Considering the lepton flavor mixing matrix V and its effective counterpart $\tilde{V}$ in matter, we have carried out a systematic analysis of their corresponding unitarity triangles in the complex plane. The exact analytical relations between the moduli $|V_{\alpha i}|$ and $|\tilde{V}_{\alpha i}|$ have been derived. The sides and angles of each unitarity triangle have also been calculated by taking account of the terrestrial matter effect. We have illustrated the shape evolution of six effective triangles with the beam energy of neutrinos and antineutrinos in a realistic long-baseline oscillation experiment. Matter corrections to the rephasing-invariant parameter of CP violation and the off-diagonal asymmetries of V have been discussed too.

We expect that this complete geometric description of matter-modified lepton flavor mixing and CP violation will be very useful for the future long-baseline neutrino oscillation experiments, although it remains unclear how far our experimentalists will go in this direction. We admit that how to experimentally realize the proposed ideas and methods is certainly a challenging question. However, it is absolutely clear that the measurement of leptonic CP violation must be a top task of experimental neutrino physics in the coming years or even decades. Let us recall what has happened in the quark sector: CP violation and one unitarity triangle (defined by $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ in the complex plane) have been established at KEK-B and SLAC-B factories, and a further study of the other five triangles will be implemented at LHC-B in the near future. Thus we are convinced that similar steps will be taken for an experimental determination of leptonic CP violation and unitarity triangles in the era of *precision* neutrino physics.

Acknowledgements. This work was supported in part by the National Nature Science Foundation of China.

Appendix A:

In the assumption of a constant earth density profile and with the help of the effective Hamiltonians given in (6), one may calculate the matter-corrected neutrino masses $\tilde{m}_i$ in an analytically exact way. The relevant results have been presented in [10,11], from which both $\widetilde{\Delta}_{ij} \equiv \tilde{m}_i^2 - \tilde{m}_j^2$ and $\widehat{\Delta}_{ij} \equiv m_i^2 - \tilde{m}_j^2$ appearing in (19)–(22) and (29)–(34) can straightforwardly be read off. Explicitly, we have

$$\begin{aligned} \widetilde{\Delta}_{21} &= \frac{2}{3}\sqrt{x^2-3y}\sqrt{3\left(1-z^2\right)} \ , \\ \widetilde{\Delta}_{31} &= \frac{1}{3}\sqrt{x^2-3y}\left[3z + \sqrt{3\left(1-z^2\right)}\right] \ , \\ \widetilde{\Delta}_{32} &= \frac{1}{3}\sqrt{x^2-3y}\left[3z - \sqrt{3\left(1-z^2\right)}\right] \ ; \end{aligned} \quad \text{(A.1)}$$

and

$$\begin{aligned} \widehat{\Delta}_{11} &= -\frac{1}{3}x + \frac{1}{3}\sqrt{x^2-3y}\left[z + \sqrt{3\left(1-z^2\right)}\right] \ , \\ \widehat{\Delta}_{22} &= -\frac{1}{3}x + \frac{1}{3}\sqrt{x^2-3y}\left[z - \sqrt{3\left(1-z^2\right)}\right] + \Delta_{21} \ , \\ \widehat{\Delta}_{33} &= -\frac{1}{3}x - \frac{2}{3}z\sqrt{x^2-3y} \ + \Delta_{31} \ ; \end{aligned} \quad \text{(A.2)}$$

together with

$$\begin{aligned} \widehat{\Delta}_{21} &= \Delta_{21} + \widehat{\Delta}_{11} \ , \\ \widehat{\Delta}_{31} &= \Delta_{31} + \widehat{\Delta}_{11} \ , \\ \widehat{\Delta}_{32} &= \Delta_{32} + \widehat{\Delta}_{22} \ , \end{aligned} \quad \text{(A.3)}$$

where $\Delta_{ij} \equiv m_i^2 - m_j^2$, and x, y and z are given by

$$\begin{aligned} x &= \Delta_{21} + \Delta_{31} + A \ , \\ y &= \Delta_{21}\Delta_{31} + A\left[\Delta_{21}\left(1-|V_{e2}|^2\right) + \Delta_{31}\left(1-|V_{e3}|^2\right)\right] \ , \\ z &= \cos\left[\frac{1}{3}\arccos\frac{2x^3 - 9xy + 27A\Delta_{21}\Delta_{31}|V_{e1}|^2}{2\left(x^2-3y\right)^{3/2}}\right] , \end{aligned} \quad \text{(A.4)}$$

with $A \equiv 2Ea$ being the matter parameter.

References

1. SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. **87**, 071301 (2001); **89**, 011301 (2002); **89**, 011302 (2002)
2. Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Lett. B **467**, 185 (1999); S. Fukuda et al., Phys. Rev. Lett. **85**, 3999 (2000); **86**, 5656 (2001)
3. KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. **90**, 021802 (2003)
4. CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B **420**, 397 (1998); Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. **84**, 3764 (2000)
5. K2K Collaboration, M.H. Ahn et al., Phys. Rev. Lett. **90**, 041801 (2003)
6. C. Jarlskog, Phys. Rev. Lett. **55**, 1839 (1985); H. Fritzsch, Z.Z. Xing, Nucl. Phys. B **556**, 49 (1999)
7. Z.Z. Xing, Int. J. Mod. Phys. A **19**, 1 (2004)

8. H. Fritzsch, Z.Z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000); hep-ph/9912358
9. Y. Farzan, A.Yu. Smirnov, Phys. Rev. D **65**, 113001 (2002)
10. V. Barger, K. Whisnant, S. Pakvasa, R.J. Phillips, Phys. Rev. D **22**, 2718 (1980); H.W. Zaglauer, K.H. Schwarzer, Z. Phys. C **40**, 273 (1988); T.K. Kuo, J. Pantaleone, Rev. Mod. Phys. **61**, 937 (1989)
11. Z.Z. Xing, Phys. Lett. B **487**, 327 (2000); Phys. Rev. D **64**, 073014 (2001)
12. L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); S.P. Mikheyev, A.Yu. Smirnov, Yad. Fiz. (Sov. J. Nucl. Phys.) **42**, 1441 (1985)
13. I. Mocioiu, R. Shrock, Phys. Rev. D **62**, 053017 (2000); and references therein
14. K. Kimura, A. Takamura, H. Yokomakura, Phys. Lett. B **537**, 86 (2002); Phys. Rev. D **66**, 073005 (2002)
15. Z.Z. Xing, Phys. Rev. D **64**, 033005 (2001)
16. P.F. Harrison, W.G. Scott, Phys. Lett. B **535**, 229 (2002); P.F. Harrison, W.G. Scott, T.J. Weiler, Phys. Lett. B **565**, 159 (2003)
17. Z.Z. Xing, Phys. Rev. D **65**, 113010 (2002)
18. V.A. Naumov, Int. J. Mod. Phys. D **1**, 379 (1992)
19. J.A. Aguilar-Saavedra, G.C. Branco, Phys. Rev. D **62**, 096009 (2000)
20. See, e.g., V. Barger, S. Geer, R. Raja, K. Whisnant, Phys. Rev. D **62**, 013004 (2000); M. Freund, M. Lindner, S.T. Petcov, A. Romanino, Nucl. Phys. B **578**, 27 (2000); and references therein
21. H. Minakata, H. Nunokawa, Nucl. Instrum. Meth. A **472**, 421 (2000); Z.Z. Xing, Phys. Rev. D **63**, 073012 (2001)
22. J. Sato, Nucl. Instrum. Meth. A **472**, 434 (2000)
23. H. Zhang, Z.Z. Xing, in preparation
24. Particle Data Group, S. Eidelman et al., Phys. Lett. B **592**, 1 (2004)